Orbits of operators

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Let $X$ be a Banach space and $T \in B(X)$. By an orbit of $T$ we mean a sequence \( \{T^n x : n = 0, 1, \ldots\} \) where $x \in X$ is a fixed vector.

Similarly, a weak orbit of $T$ is a sequence \( \{\langle T^n x, x^* \rangle : n = 0, 1, \ldots\} \) and a polynomial orbit of $T$ is a set of the form \( \{p(T)x : p \text{ polynomial}\} \) where $x \in X$ and $x^* \in X^*$.

The definition comes from the theory of dynamical systems. In the context of operator theory the notion was first used by Rolewicz [Ro]. Orbits of operators are closely connected with the local spectral theory, the theory of semigroups of operators [N], and especially, with the invariant subspace problem, see e.g. [B]. Indeed, an operator $T \in B(X)$ has no nontrivial closed invariant subspace if and only if all orbits corresponding to nonzero vectors span all the space $X$.

For simplicity we consider only complex Banach spaces. However, many results remain true also for real Banach spaces.

1. Orbits

Typically, the behaviour of an orbit \( \{T^n x : n = 0, 1, \ldots\} \) depends much on the vector $x \in X$. An operator can have some orbits very regular and other orbits extremely irregular.

Example 1. Let $T = 2S$ where $S$ is the backward shift on a separable Hilbert space $H$. Then:
(i) there is a dense subset $M_1 \subset H$ such that $T^n x \to 0$ ($x \in M_1$);
(ii) there is a dense subset $M_2 \subset H$ such that $\|T^n x\| \to \infty$ ($x \in M_2$);
(iii) there is a residual subset $M_3 \subset H$ such that the set \( \{T^n x : n = 0, 1, \ldots\} \) is dense in $H$ for all $x \in M_3$.

The existence of regular orbits is given by the following result [M1], [B]:

Theorem 2. Let $X$ be a Banach space, let $T \in B(X)$, $\varepsilon > 0$ and let $(a_n)$ be a sequence of positive numbers such that $\lim a_n = 0$. Then:
(i) there exists a vector $x \in X$ such that $\|x\| < \sup a_n + \varepsilon$ and $\|T^n x\| \geq a_n \cdot r(T^n)$ for all $n$;
(ii) there exists a dense subset of points $x \in X$ such that $\|T^n x\| \geq a_n r(T^n)$ for all but a finite number of $n$’s.

Corollary 3. Let $T \in B(X)$. Then:
(i) the set \( \{x \in X : \liminf \|T^n x\|^{1/n} = r(T)\} \) is dense;
(ii) the set \( \{x \in X : \limsup \|T^n x\|^{1/n} = r(T)\} \) is residual;
(iii) the set of all $x \in X$ such that the limit $\lim \|T^n x\|^{1/n}$ exists (and is equal to $r(T)$) is dense.
Recall that the quantity \( \limsup \|T^n x\|^{1/n} \) is called the local spectral radius of \( T \) at \( x \). It plays an important role in the local spectral theory.

As another corollary we get that the infimum and the supremum in the spectral radius formula

\[
r(T) = \inf_k \|T^k\|^{1/k} = \inf_k \|T^k x\|^{1/k}
\]

can be exchanged.

**Corollary 4.** Let \( T \in B(X) \). Then

\[
r(T) = \inf_k \|T^k x\|^{1/k} = \sup_{\|x\|=1} \inf_k \|T^k x\|^{1/k}.
\]

It is also possible to replace the spectral radius \( r(T^n) \) by the norm \( \|T^n\| \) in Theorem 2. In this case, however, there is a restriction on the sequence \((a_n)\).

**Theorem 5.** Let \( T \in B(X) \), let \((a_n)\) be a sequence of positive numbers such that \( \sum a_n^{2/3} < \infty \). Then there exists \( x \in X \) such that \( \|T^n x\| \geq a_n \|T^n\| \) for all \( n \).

The following result is true for Hilbert space operators; in Banach spaces it is false.

**Theorem 6.** Let \( T \) be a non-nilpotent operator on a Hilbert space \( H \). Then the set \( \{ x \in H : \sum \frac{\|T^n x\|}{\|T^n\|} = \infty \} \) is residual.

## 2. Hypercyclic vectors

Vectors with extremely irregular orbits are called hypercyclic. More precisely, \( x \in X \) is hypercyclic for an operator \( T \in B(X) \) if the set \( \{T^n x : n = 0, 1, \ldots\} \) is dense in \( X \).

Recall also that a vector \( x \in X \) is called cyclic for \( T \in B(X) \) if the set \( \{p(T)x : p \text{ polynomial}\} \) is dense in \( X \) and supercyclic for \( T \) if \( \{\lambda T^n x : \lambda \in \mathbb{C}, n = 0, 1, \ldots\}^{-} = X \).

The basic properties of hypercyclic vectors are summarized below:

**Proposition 7.** Let \( T \in B(X) \), let \( x \in X \) be hypercyclic for \( T \). Then:

(i) there is a residual set of vectors hypercyclic for \( T \);

(ii) \( \sigma_p(T^*) = \emptyset \), where \( \sigma_p \) denotes the point spectrum;

(iii) \( p(T)x \) is hypercyclic for \( T \) for each nonzero polynomial \( p \);

(iv) \( x \) is hypercyclic for \( T^n \) for each \( n \).

(v) \( x \) is hypercyclic for \( \alpha T \) for each \( \alpha \in \mathbb{C} \), \(|\alpha| = 1\).

The following criterion provides a simple way of constructing hypercyclic vectors, see e.g. [GS]:

**Theorem 8.** Let \( T \in B(X) \). Suppose that there is an increasing sequence of positive integers \((n_k)\) such that:
(i) there is a dense subset $X_0 \subset X$ such that $\lim_{k \to \infty} T^{n_k} x \to 0$ for all $x \in X_0$;
(ii) $\bigcup_k T^{n_k} B_X$ is dense in $X$, where $B_X$ denotes the closed unit ball in $X$.

Then there exists a vector hypercyclic for $T$. By Proposition 7, this means that the set of all hypercyclic vectors is residual.

Although it is relatively easy to construct an operator with a residual set of hypercyclic vectors (see Example 1), it is extremely difficult to construct an operator with all nonzero vectors hypercyclic. The first example of this type was constructed by Read [R] on $\ell_1$. Equivalently, such an operator has no nontrivial closed invariant subset. It is an open problem whether this can happen in Hilbert spaces.

3. Weak orbits

Weak orbits were introduced and first studied by van Neerven. Many results for orbits of operators can be modified also for weak orbits. For a survey of results see e.g. [N], [M4].

**Theorem 9.** Let $T \in B(X)$, let $a_n > 0$, $\sum a_n^{1/3} < \infty$. Then there exist $x \in X$ and $x^* \in X^*$ such that $|\langle T^n x, x^* \rangle| \geq a_n \|T^n\|$ for all $n$.

**Theorem 10.** Let $T \in B(X)$. Then:
(i) the set $\{ (x, x^*) \in X \times X^* : \liminf |\langle T^n x, x^* \rangle|^{1/n} = r(T) \}$ is dense;
(ii) the set $\{ (x, x^*) \in X \times X^* : \limsup |\langle T^n x, x^* \rangle|^{1/n} = r(T) \}$ is residual;
(iii) the set of all pairs $(x, x^*) \in X \times X^*$ such that the limit $\lim |\langle T^n x, x^* \rangle|^{1/n}$ exists (and is equal to $r(T)$) is dense.

The statement analogous to Theorem 2 for weak orbits is an open problem:

**Problem 11.** Let $T \in B(X)$, let $(a_n)$ be a sequence of positive numbers satisfying $\lim a_n = 0$. Do there exists $x \in X$ and $x^* \in X^*$ such that $|\langle T^n x, x^* \rangle| \geq a_n r(T^n)$ for all $n$?

The statement is false for real Banach spaces. A partial positive answer is given in the following case which is important from the point of view of the invariant subspace problem.

**Theorem 12.** Let $T$ be an operator on a Hilbert space $H$ such that $r(T) = 1$ and $T^n x \to 0$ for all $x \in H$. Let $(a_n)$ be a sequence of positive numbers satisfying $\lim a_n = 0$. Then there exists $x \in H$ such that $|\langle T^n x, x \rangle| > a_n$ for all $n$.

The statement analogous to Theorem 6 for weak orbits is not true:

**Theorem 13.** There exists an operator $T \in B(H)$ such that $\sum \frac{|\langle T^n x, y \rangle|}{\|T^n\|} < \infty$ for all $x, y \in H$. 

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4. Scott Brown technique

The Scott Brown technique is an efficient way of constructing invariant subspaces. It was first used for subnormal operators but later it was adopted for contractions on Hilbert spaces. Some results are known also for Banach space operators and even for $n$-tuples of commuting operators.

The basic idea of the Scott Brown technique is to construct a weak orbit $\{⟨T^n x, x^*⟩: n = 0, 1, \ldots\}$ which behaves in a precise way. Typically, vectors $x \in X$ and $x^* \in X^*$ are constructed such that

$$\langle T^n x, x^* \rangle = \begin{cases} 0 & n \geq 1; \\ 1 & n = 0. \end{cases}$$

Then $T$ has a nontrivial closed invariant subspace. Indeed, either $Tx = 0$ (and $x$ generates a 1-dimensional invariant subspace) or $\{T^n x : n \geq 1\}$ generates a nontrivial closed invariant subspace.

The best result obtained by the Scott Brown technique is [BCP]:

**Theorem 14.** Let $T$ be an operator on a Hilbert space $H$ such that $\|T\| \leq 1$ and $\sigma(T) \supset \{z : |z| = 1\}$. Then $T$ has a nontrivial closed invariant subspace.

Using Theorem 12 and techniques of Proposition 7 (v) it is possible to obtain the following result formally analogous to Theorem 14:

**Theorem 15.** Let $T \in B(H)$ be a power bounded operator satisfying $r(T) = 1$. Then there is a nonzero vector $x \in H$ such that $x$ is not supercyclic.

5. Polynomial orbits

If $x$ is an eigenvalue of $T$, $Tx = \lambda x$ for some complex $\lambda$, then $p(T)x = p(\lambda)x$ for every polynomial $p$, and so we have a complete information about the polynomial orbit $\{p(T)x : p \text{ polynomial}\}$. Unfortunately, operators on infinite dimensional Banach spaces have usually no eigenvalues. The proper tool appears to be the notion of the essential approximate point spectrum of $T$.

Denote by $\sigma_e(T)$ the essential spectrum of $T \in B(X)$, i.e., the spectrum of $\rho(T)$ in the Calkin algebra $B(X)/\mathcal{K}(X)$, where $\mathcal{K}(X)$ is the ideal of all compact operators on $X$ and $\rho : B(X) \to B(X)/\mathcal{K}(X)$ is the canonical projection. Denote further by $\sigma_{pe}(T)$ the essential approximate point spectrum of $T$, i.e., $\sigma_{pe}(T)$ is the set of all complex $\lambda$ such that

$$\inf\{\|(T - \lambda)x\| : x \in M, \|x\| = 1\} = 0$$

for every subspace $M \subset X$ with $\text{codim} \ M < \infty$.

It is easy to see that $\lambda \notin \sigma_{pe}(T)$ if and only if $\dim \ker (T - \lambda) < \infty$ and $T - \lambda$ has closed range, i.e., if $T - \lambda$ is upper semi-Fredholm. It is known [HW] that $\sigma_{pe}(T)$ contains the topological boundary of the essential spectrum $\sigma_e(T)$. In particular, $\sigma_{pe}(T)$ is a nonempty compact subset of the complex plane for each $T$ on an infinite dimensional Banach space $X$. 


Theorem 16. Let $T \in B(X)$, $\lambda \in \sigma_{\pi e}(T)$. Let $(a_n)$ be a sequence of positive numbers with $\lim_{k \to \infty} a_k = 0$. Then:

(i) there exists $x \in X$ such that
\[ \|p(T)x\| \geq a_{\deg p} \cdot |p(\lambda)| \]
for every polynomial $p$;

(ii) let $u \in X$, $\varepsilon > 0$. Then there exists $y \in X$ and a positive constant $C = C(\varepsilon)$ such that $\|y - u\| \leq \varepsilon$ and
\[ \|p(T)y\| \geq C \cdot a_{\deg p} \cdot |p(\lambda)| \]
for every polynomial $p$.

In the previous theorem we expressed the estimate of $\|p(T)x\|$ by means of $|p(\lambda)|$ where $\lambda$ was a fixed element of $\sigma_{\pi e}(T)$. Next we are looking for an estimate in terms of $\max\{|p(\lambda)| : \lambda \in \sigma_{\pi e}(T)\}$. Since $\partial \sigma_{\pi e}(T) \supset \sigma_{\pi e}(T)$, by the spectral mapping theorem for $\sigma_{\pi e}$ we have
\[ \max_{\lambda \in \sigma_{\pi e}(T)} |p(\lambda)| = \max_{\lambda \in \sigma_{\pi e}(T)} |p(\lambda)| = \max\{|z| : z \in \sigma_{\pi e}(p(T))\} \]
where $r_e$ denotes the essential spectral radius.

An important tool for the results in this section is the following classical lemma of Fekete [F]:

Lemma 17. Let $K$ be a non-empty compact subset of the complex plane and let $k \geq 1$. Then there exist points $u_0, u_1, \ldots, u_k \in K$ such that
\[ \max\{|p(z)| : z \in K\} \leq (k + 1) \cdot \max_{0 \leq i \leq k} |p(u_i)| \]
for every polynomial $p$ with $\deg p \leq k$.

By using the previous lemma we can get [M2], [M3]

Proposition 18. Let $T \in B(X)$, $\varepsilon \geq 0$ and $k \geq 1$. Then:

(i) if $\text{card} \sigma_{\pi e}(T) \geq k + 1$ then there exists $x \in X$ with $\|x\| = 1$ and
\[ \|p(T)x\| \geq \frac{1 - \varepsilon}{k + 1} r_e(p(T)) \]
for every polynomial $p$ with $\deg p \leq k$.

(ii) let $x \in X$ and $\varepsilon > 0$. Then there exists $y \in X$ and a positive constant $C = C(\varepsilon)$ such that $\|y - x\| \leq \varepsilon$ and
\[ \|p(T)y\| \geq C \cdot (1 + \deg p)^{-{(1+\varepsilon)}} r_e(p(T)) \]
for every polynomial $p$. 

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6. Capacity

The notion of capacity of an operator was defined by Halmos [H]. If \( T \in B(X) \) then

\[
\text{cap} T = \lim_{k \to \infty} (\text{cap}_k T)^{1/k} = \inf_k (\text{cap}_k T)^{1/k}
\]

where

\[
\text{cap}_k T = \inf \{ \| p(T) \| : p \in \mathcal{P}_k^1 \}
\]

and \( \mathcal{P}_k^1 \) is the set of all monic (i.e., with leading coefficient equal to 1) polynomials of degree \( k \).

This is a generalization of the classical notion of capacity of a compact subset \( K \) of the complex plane:

\[
\text{cap} K = \lim_{k \to \infty} (\text{cap}_k K)^{1/k} = \inf_k (\text{cap}_k K)^{1/k}
\]

where

\[
\text{cap}_k K = \inf \{ \| p \|_K : p \in \mathcal{P}_k^1 \} \quad \text{and} \quad \| p \|_K = \sup \{ |p(z)| : z \in K \}.
\]

By the main result of [H], \( \text{cap} T = \text{cap} \sigma(T) \).

The local capacity of \( T \) at \( x \) can be defined analogously:

\[
\text{cap}_k (T, x) = \inf \{ \| p(T)x \| : p \in \mathcal{P}_k^1 \}
\]

and

\[
\text{cap} (T, x) = \limsup_{k \to \infty} \text{cap}_k (T, x)^{1/k}
\]

(in general the limit does not exist).

It is easy to see that \( \text{cap}(T, x) \leq \text{cap} T \) for every \( x \in X \).

**Theorem 19.** Let \( T \in B(X) \). Then the set \( \{ x \in X : \text{cap}(T, x) = \text{cap} T \} \) is residual in \( X \).

An operator \( T \in B(X) \) is called quasialegbraic if and only if \( \text{cap} T = 0 \). Similarly, \( T \) is called locally quasialegbraic if \( \text{cap}(T, x) = 0 \) for every \( x \in X \).

It follows from Theorem 19 that these two notions are equivalent.

**Theorem 20.** An operator is quasialegbraic if and only if it is locally quasialegbraic.

Theorem 20 is an analogy to the well-known result of Kaplansky: an operator is algebraic (i.e. \( p(T) = 0 \) for some non-zero polynomial \( p \)) if and only if it is locally algebraic (i.e. for every \( x \in X \) there exists a polynomial \( p_x \neq 0 \) such that \( p_x(T)x = 0 \)).
References


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